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A Characterization Of Geometry Of Hyperbolic Type

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الملخص

نقدم في هذا البحث نظرية تحوي شروطا لوصف هندسة نقطة-خط من نوع $D_{n,k}$ ، حيث $n \geq k+3$. $k \geq 2$

ABSTRACT

We give in this paper a theorem which characterizes, by axioms on points and lines, a point –line geometry of type $D_{n,k}$ where $k \geq 2$ and $n \geq k+3$.

Keyword: parapolar space, Symplecton, building geometry.

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1- Introduction

In [1] Veblen and Young in their classical work characterize $A_{n,1}$ in terms of points and lines. Polar spaces were characterized by Buekenhout-Shult [2]. In [3] Lie incidence systems of type $F_{4,1}$ are characterized as parapolar spaces. Some incidence structures associated with maximal parabolic representations of groups of Lie type are characterized by Cooperstein and Cohen, see [4] and [5]. The building of type $C_{3,2}$ and the general case $C_{n,n-1}$ are characterized in Lehman [6]. In [7] the author was the first who has studied the geometry of type $D_{n,k}$, where he presented the structure of such geometry . Here we will use axioms to characterize the geometry $D_{n,k}$. First we present some definition of terminology's that will be used.

A point-line geometry Γ=(P,L) is an incident system with a non-empty set P of points to gether with a collection L of subsets of cardinality ≥ 2 called lines.

The singular rank of a space Γ is the maximal number *n* (possibly ∞) for which there exist a chain of distinct subspaces $\varphi \neq X_0 \subset X_1 \subset \ldots \subset X_n$ such that X_i is singular for each *i*, $X_i \neq X_j$, $i \neq j$. For example rank(φ)=-1, rank($\{p\}$)=0 where *p* is a point and rank(*l*)=1 where *l* a line.

$x^{\perp}$ means the set of all points in P collinear with x, including *x* itself.

A subspace of a point-line geometry Γ=(P, L) is a subset $X \subseteq P$ such that any line which has at least two of its incident points in X has all of its incident points in X. <X> means the intersection over all subspaces containing X, where $X \subseteq P$. Lines incident with more than two points are called *thick* lines, those incident with exactly two points are called *thin* lines. ***A geometric hyperplane*** is a proper subspace H of Γ such that each line of L intersects H non-trivially.

In a point-line geometry Γ=(P, L), a path of length n is a sequence of n+1 points $(x_0,x_1,..,x_n)$ where, (x_i,x_{i+1}) are collinear, x_0 is called the initial point and x_n is called the end point. ***A geodesic*** from a point x to a point y is a path of minimal possible length with initial point *x* and end point *y.* We denote this length by d_{Γ} (x, y), the length of the geodesic from x to y is called the distance between x and y.

The diameter of the geometry is the maximal distance of points.

A point-line geometry Γ is called ***connected*** if and only if for any two of its points there is a path connecting them. A subset X of P is said to be ***convex*** if X contains all points of all geodesics connecting two points of X.

A polar space is a point-line geometry Γ=(P, L) satisfying the Buekenhout-Shult axiom [2] :
For each point-line pair (*p, l*) with *p* not incident with *l*; *p* is collinear with one or all points of *l*, that is $|p^{\perp}\cap l|=1$ ($|p^{\perp}\cap l|$ means the number of points of *l* that are collenear to the point *p*) or else $p^{\perp}\supset l$. Clearly this axiom is equivalent to saying that $p^{\perp}$ is a geometric hyperplane of Γ for every point $p\in$P.

A point-line geometry Γ=(P, L) is called ***a projective plane*** if and only if it satisfies the following conditions [2]:

(i) Γ is a linear space; every two distinct points x, y in P lie exactly on one line,

(ii) every two lines intersect in one point,

(iii) there are four points no three of them are on a line.

A point-line geometry Γ=(P, L) is called ***a projective space*** if the following conditions are satisfied:

(i) every two points in P lie exactly on one line ,

(ii) if l_1, l_2 are two lines in L and $l_1\cap l_2\neq\emptyset$, then $\langle l_1, l_2\rangle$ is a projective plane. ($\langle l_1, l_2\rangle$ means the smallest subspace of Γ containing l_1 and l_2.)

A gamma space is a point-line geometry Γ=(P, L) satisfying the axiom [2] :
For each point-line pair (*p, l*) with *p* not incident with *l*; $p^{\perp}\cap l$ is either empty or one point of *l* or else $p^{\perp}\supset l$.

A point-line geometry Γ=(P, L) is called ***a parapolar*** *space* if and only if it satisfies the following properties:

(i) Γ is a connected gamma space,

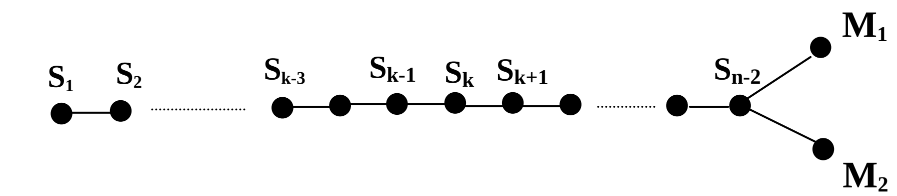
(ii) for every line *l*; $l^{\perp}$ s not a singular subspace,

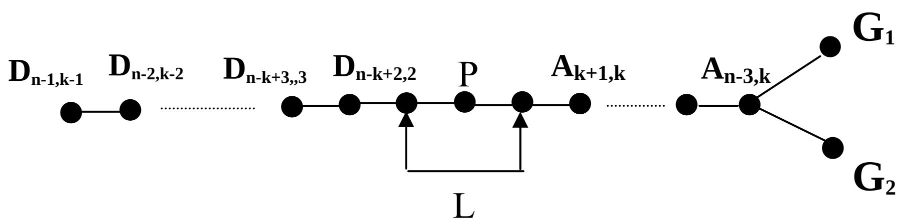
(iii) for every pair of non-collinear points x, y; $x^{\perp}\cap y^{\perp}$ is either empty, a single point, or a non-degenerate polar space of rank at least 2 [2].

If x, y are distinct points in P, and if $|x^{\perp}\cap y^{\perp}|=1$, then (x, y) is called ***a special pair***, and if $x^{\perp}\cap y^{\perp}$ is a polar space, then (x, y) is called ***a polar pair*** (or ***a symplectic pair***). A parapolar space is called ***a strong parapolar*** space if it has no special pairs [4].

2. Construction of $D_{n,k}$ [7]

Consider the polar space $\Delta=\Omega^{+}(V_{2n}, F)$ that comes from a vector space V of dimension 2n over a finite field F=GF(*q*) with a symmetric hyperbolic bilinear form B. $\mathbf{S_i}$ is the set of all totally isotropic i-dimensional subspaces of V*;* $1\leq i\leq n-2$. The two classes $\mathbf{M_1}$, $\mathbf{M_2}$ consist of maximal totally isotropic n-dimensional subspaces. Two n-subspaces fall in the same class if their intersection is of odd dimension.





$\mathbf{D_{n,k}(F)}$

The geometry of type $\mathbf{D_{n,k}(F)}$ is the point-line geometry (P, L), whose set of points P is the collection of all k-dimensional subspaces of the vector space V, and whose lines are the pairs (A, B) where A is (k-1)-dimensional subspace of (k+1)-subspace B (A is a subspace of dimension k-1 , B is a subspace of dimension k+1 and A is a subspace of B)–that is, the set of (k-1, k+1)-subspace of V. A point C is incident with a line (A, B) if and only if $A\subset C\subset B$ as subspaces of V.

To define the collinearity, let C_1 and C_2 be two points (the points are the totally isotropic k-spaces), then C_1 is collinear to C_2 if and only if the intersection of $C_1\cap C_2$=(k-1)-space and $\langle C_1, C_2\rangle$=(k+1)-space.

The elements of the classes $\mathbf{G_1}$ and $\mathbf{G_2}$ are Grassmannian geometries of type $\mathbf{A_{n-1,k}}$.

There are two kinds of symplecta (1) The first kind is the convex polar spaces $A_{3,2}$ that represent the (k-2, k+2) subspaces of V. Then symplecton S of kind $A_{3,2}$ is the set of totally isotropic k- spaces that contain the totally isotropic (k-2)-dimensional space and contained in the totally isotropic (k+2)-dimensional space. (2) The second kind of symplecta is the convex polar spaces of type $D_{n-k+1,1}$ that represent the collection of all totally isotrtopic (k-1)-subspaces of V. Thus this kind of symplecta is defined as the collection of all totally isotropic k-subspaces of V that contain such totally isotropic (k-1)-spaces.

Remark. Each geometry of the class G_1 or G_2 of $D_{n,k}$ is denoted by A_T, where T is the totally isotropic n-space that corresponds to such geometry and A_D denotes to the sypmlecton of type $A_{3,2}$, where D is the totally isotropic (k+2)-space that corresponds to such symplecton.

3- The main result

To make a characterization of the geometry of type $D_{n,k}$ we prove the following theorem:

MAIN THEOREM.

1- Let Γ=(P,L) be a space of point line geometry satisfying the following axioms:

(P_1) Γ is a weak parapolar space.

(P_2) There exist two kinds of symplecta of rank at least 2.

(P_3) If (x, S) is a pair of non-incident point-symplecton, then rank($x^{\perp}\cap S$)=-1,1,2.

(P_4) If S_1 and S_2 are two different symplecta of $D_{n,k}$, then rank($S_1\cap S_2$)=-1, 0, 2.

Then Γ is a point-line geometry type $D_{n,k}$ ($k\geq 2$, $n\geq k+3$).

2- If Γ is a point-line geometry of type $D_{n,k}$ ($k\geq 2$, $n\geq k+3$), then (P_1), (P_2) and (P_3) are satisfied.

To prove the main theorem, the following Lemmas must be proved:

3.1 Lemma. The point line geometry of type $D_{n,k}$ is a weak parapolar space.

Proof: By using building diagram of $D_{n,k}$ (page 5), we see that the diagram geometry of type $D_{n,2}$ is a sub-geometry of $D_{n,k}$. It has been proved in [8] that $D_{n,2}$ is a weak geometry, so is $D_{n,k}$. To prove that the geometry is parapolar, we first prove that $D_{n,k}$ is gamma space. Assume that *l* is a line containing two points p and q such that $\psi(p)=\langle x_1,x_2,\ldots,x_k\rangle$ and $\psi(q)=\langle y_1,y_2,\ldots,y_k\rangle$. Then by the definition of collinearity, the two points form a totally isotropic (k+1)-space $\langle z_1,z_2,\ldots,z_{k-1},x_1,y_1\rangle$, where $z_1=x_2=y_2$, $z_2=x_3=y_3,\ldots, z_{k-1}=x_k=y_k$. Let s be a point not in *l* such that s is collinear to p and q, we shall prove that s is collinear to every point incident to *l*. Since s is collinear to p and q, then s does not contain the totally isotropic (k-1)-space $\langle z_1,z_2,\ldots,z_{k-1}\rangle$ and form (k+1)-spaces with p and q. Now assume that r is arbitrary point in *l* such that $r\neq p\neq q$, then $\psi(r)$, $\psi(s)$ contain the same (k-1)-

space $\langle z_1,z_2,\ldots,z_{k-1}\rangle$ and they are contained in (k+1)-space $\langle z_1,z_2,\ldots,z_{k-1},x_1,y_1\rangle$. Thus s is collinear to r.

Now for the completion of the proof we need to show that for any line l ; $l^{\perp}$ is non-singular. For this purpose we take the same above line i.,e., $l=(\langle z_1,z_2,\ldots,z_{k-1}\rangle, \langle z_1,z_2,\ldots,z_{k-1},x_1,y_1\rangle)$. Thus the points r, s can be chosen to be non-collinear and each of them is collinear to p and q. Since $\psi(s)$ is contained in a maximal n-space, then we take $\langle z_1,z_2,\ldots,z_{k-2},x_1,y_1\rangle$, $\langle z_1,z_2,\ldots,z_{k-1},u\rangle$ to the points r, s where u is a vector in the n-space not in $\psi(s)$. Now $\langle z_1,z_2,\ldots,z_{k-2},x_1,y_1\rangle\cap\langle z_1,z_2,\ldots,z_{k-1},u\rangle$ does not contain a (k-1)-space, this means that r is not collinear to s but $\langle z_1,z_2,\ldots,z_{k-1},x_1,y_1\rangle$ and $\langle z_1,z_2,\ldots,z_{k-1},y_1,u\rangle$ form (k+1)-spaces, then r, s are collinear to each of p and q. Thus $l^{\perp}$ is non-singular. ♦

3.2 Lemma. Let (p, S) be a non-incident pair of point and symplecton in $D_{n,k}$, then rank($p^{\perp}\cap S$)=-1,1,2.

Proof.

a- For any pair (p, A_D) of a point p and a symplecton A_D of type $A_{3,2}$, there are two cases:

ai- If dim($\psi(p)\cap D$)<k-1 (D is the the totally isotropic subspace that corresponds to a symplecton A_D), then there is no any k-space contained in D and meets $\psi(p)$ in a (k-1)-space i.e., $\psi(p)\cap D=\varphi$ and rank($p^{\perp}\cap S$)=-1.

aii- If dim($\psi(p)\cap D$)=k-1, then se t of k-spaces in D that contain the (k-1)-space $\psi(p)\cap D$ form a projective plane i.e., rank($p^{\perp}\cap S$)=2.

b- For any pair (p, S) of a point p and a symplecton S of type $D_{n-k+1,1}$, there are two cases:

bi- If dim(($\psi(p)\cap\psi(S)$)<k-2, then we cannot find any k-space contains $\psi(S)$ and meets $\psi(p)$ in a (k-1)-space i.e., rank($p^{\perp}\cap S$)=-1.

bii- If dim(($\psi(p)\cap\psi(S)$)=k-2, and $\langle\psi(p),\psi(S)\rangle$=(k+1)-space, then there are two k-spaces meet $\psi(p)$ in a (k-1)-space and contain $\psi(S)$. Since the two k-space form a (k+1)-space, then $p^{\perp}\cap S$ is a line i.e., rank($p^{\perp}\cap S$)=1

3.3 Lemma. Let S_1 and S_2 be two distinct symplecta of $D_{n,k}$, then rank($S_1\cap S_2$)=-1,0, 2.

Proof. We have three cases:

1- Let A_{D_1}, A_{D_2} be two distinct symplecta of type $A_{3,2}$ then rank($A_{D_1}\cap A_{D_2}$)=-1,0,2.

2- If S_1 and S_2 are symplecta of type $D_{n-1,1}$, then rank($S_1\cap S_2$)=-1, 0.

3- If S is a symplecton of type $D_{n-1,1}$ and A_D is a symplecton of type $A_{3,2}$, then rank($S\cap A_D$)=-1, 2.

To prove **1** let A_{T_1} and A_{T_2} be two spaces located in the same class (**G_1**), so A_{D_1} and A_{D_2} are located in the located in **G_1,** then we have two cases:

a- A_{D_1}, A_{D_2} are symplecta of the same geometry A_T, then it has three cases:

a1- If $\dim(D_1 \cap D_2) \leq (k\text{-}1)$-space, then $\mathrm{rank}(A_{D_1} \cap A_{D_2})=-1$.

a2- if $\dim(D_1 \cap D_2)=k$-space, then $\mathrm{rank}(A_{D_1} \cap A_{D_2})=0$.

a3- if $\dim(D_1 \cap D_2)=(k+1)$-space, then $\mathrm{rank}(A_{D_1} \cap A_{D_2})=2$.

b- A_{D_1}, A_{D_2} are symplecta of different geometries A_{T_1} and A_{T_2} respectively, then $T_1 \cap T_2$ is a space of odd dimension and we have three cases:

b1- For $\dim(T_1 \cap T_2) \leq k\text{-}1$, $\dim(D_1 \cap D_2) \leq k\text{-}1$, so $\mathrm{rank}(A_{D_1} \cap A_{D_2})=-1$.

b2- For $\dim(T_1 \cap T_2)=k$, $\dim(D_1 \cap D_2) \leq k$. Then $\mathrm{rank}(A_{D_1} \cap A_{D_2})=-1, 0$.

b3- For $\dim(T_1 \cap T_2)>k$, then either $\dim(D_1 \cap D_2) \leq k$ and $\mathrm{rank}(A_{D_1} \cap A_{D_2})=-1, 0$ or $\dim(D_1 \cap D_2)=k+1$ and $\mathrm{rank}(A_{D_1} \cap A_{D_2})=2$. Then $\mathrm{rank}(A_{D_1} \cap A_{D_2})=-1, 0, 2$.

Now the case in which A_{T_1} and A_{T_2} are located in different classes is similar to that case above i.e, $\mathrm{rank}(A_{D_1} \cap A_{D_2})=-1, 0, 2$.

To prove 2- $\psi(S_1)$ and $\psi(S_2)$ correspond to (k-1)-spaces. We have two cases:

i- If $\psi(S_1) \cap \psi(S_2) \subseteq (k\text{-}2)$-space, then we cannot find a k-space that contains $\langle \psi(S_1), \psi(S_2) \rangle$, i.e., $S_1 \cap S_2 = \varphi$. Thus $\mathrm{rank}(S_1 \cap S_2)=-1$.

ii- If $\psi(S_1) \cap \psi(S_2)=(k\text{-}2)$space and $\langle \psi(S_1), \psi(S_2) \rangle = k$-space, then $\mathrm{rank}(S_1 \cap S_2)=0$.

To prove 3- If S is a symplecton of type $D_{n-1,1}$ and A_D is a symplecton of type $A_{3,2}$, then we have two cases:

i- $\psi(S) \cap D \subseteq (k\text{-}2)$-space, then we there is no a k-space that contains $\psi(S)$ and contained in D, i.e., $\mathrm{rank}(S \cap A_D)=-1$.

ii- $\psi(S) \subseteq D$, then the number of different k-spaces in D that contains $\psi(S)$ form a space of rank 3 i.e., $\mathrm{rank}(S \cap A_D)=2$.♦

Proof of the main theorem.

The proofs of (p_1), (p_3) and (p_4) are exactly the proofs of Lemma 3.1, Lemma 3.2 and Lemma 3.3 respectevely. From the construction of $D_{n,k}$ it has been shown that the geometry has two kinds of symplecta The first kind is the convex polar spaces $A_{3,2}$ that represent the (k-2, k+2) subspaces of V. The second kind of symplecta is the convex polar spaces of type $D_{n-k+1,1}$ that represent the collection of all totally isotropic (k-1)-subspaces of V. Thus this kind of symplecta is defined as the collection of all TI k-subspaces of V that contain such totally isotropic (k-1)-spaces, then (p_2) is satisfied. All geometries with Coxeter diagrams of spherical type with rank n have been characterized in [4], [10] and [5], the class of strong parapolar spaces also have been characterized in [9] and throughout those papers it has been found that all geometries of types (Grrassmannin-Exceptional type-Half-spin-metasymplectic- polar grassmmann geometries of type $C_{n,k}$) have one kind of symplecton that is a convex polar space of rank at least 3. Then any geometry satisfies the axioms (p_1) - (p_4) is a building of type $D_{n,k}(F)$ for $k \geq 2$ and $n \geq k+3$.

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