Hamilton-Jacobi Quantization Of Continuous Systems With Higher-Order Lagrangian Density

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الملخص

تكميم الأنظمة المتصلة ذوات اللاجرانج عالية الدرجة بطريقة الهاميلتون جاكوبي

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ABSTRACT

Continuous systems with higher-order Lagrangian density are treated as first order Lagrangian density by using Hamilton-Jacobi method. An example is studied in details.

Keywords

Hamilton-Jacbi formalism, Higher-order Lagrangian.

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INTRODUCTION:

The study of singular systems has reached a great status in physics, since the development of the Hamiltonian formulation by Dirac [1,2]. The theories with higher order singular Lagrangian have been developed by Ostrogradskii [3].

Hamilton-Jacobi approach was developed to study singular first order systems [4-7]. The generalization of the Hamilton-Jacobi approach for higher-order singular system was developed in Refs.[8,9]. The quantization of second-order Lagrangian was discussed by Muslih [10]. In Ref. [11], the higher order effective lagrangian was reduced to a first-order one.

The main idea of this work is to apply Hamilton-Jacobi approach to the reduction form of the higher order lagrangian density. The higher order regular Lagrangian was treated as first-order singular Lagrangian for discrete system [12,13].

First, begin with second order Lagrangian density $L\left(\phi,\partial_{\mu_1}\phi,\partial_{\mu_1}\partial_{\mu_2}\phi\right)$, and convert it to first order singular Lagrangian by introducing $Z_{\mu_1}=\partial_{\mu_1}\phi$, and then $\lambda_{\mu_1}=Z_{\mu_1}-\partial_{\mu_1}\phi$.

The new form of the Lagrangian density is written as

$$L\left(\phi,\partial_{\mu_1}\phi,\partial_{\mu_1}\partial_{\mu_2}\phi\right)=L_o\left(\phi,Z_{\mu_1},\partial_{\mu_2}Z_{\mu_1}\right)+\lambda_{\mu_1}\left(Z_{\mu_1}-\partial_{\mu_1}\phi\right). \qquad (1)$$

To construct the corresponding Hamiltonian density, let us define the momenta

$$P_{\mu_1}=\frac{\partial L}{\partial\left(\partial_{\mu_1}\phi\right)}=-\lambda_{\mu_1}, \qquad (2)$$

$$P_{\mu_1,\mu_2}=\frac{\partial L}{\partial\left(\partial_{\mu_2}Z_{\mu_1}\right)}=\frac{\partial L_o}{\partial\left(\partial_{\mu_2}Z_{\mu_1}\right)}. \qquad (3)$$

Therefore, the canonical Hamiltonian density is

$$H=Z_{\mu_1}P_{\mu_1}+\left(\partial_{\mu_2}Z_{\mu_1}\right)P_{\mu_1,\mu_2}-L_o\left(\phi,Z_{\mu_1},\partial_{\mu_2}Z_{\mu_1}\right). \qquad (4)$$

The Hamilton's equations of motion are

$$\frac{\partial\phi}{\partial x_{\mu_1}}=\frac{\partial H}{\partial P_{\mu_1}}=Z_{\mu_1}, \qquad (5)$$

$$\frac{\partial Z_{\mu_1}}{\partial x_{\mu_2}}=\frac{\partial H}{\partial P_{\mu_1,\mu_2}}=\partial_{\mu_2}Z_{\mu_1}, \qquad (6)$$

$$\frac{\partial P_{\mu_1}}{\partial x_{\mu_1}} = -\frac{\partial H}{\partial \phi} = \frac{\partial L_o}{\partial \phi}, \tag{7}$$

$$\frac{\partial P_{\mu_1,\mu_2}}{\partial x_{\mu_2}} = -\frac{\partial H}{\partial Z_{\mu_1}} = \frac{\partial L_o}{\partial Z_{\mu_1}} - P_{\mu_1}. \tag{8}$$

One can solve eqs(5-8) simultaneously and get

$$\frac{\partial L_o}{\partial \phi} - \partial_{\mu_1}\left(\frac{\partial L_o}{\partial\left(\partial_{\mu_1}\phi\right)}\right) + \partial_{\mu_1}\partial_{\mu_2}\left(\frac{\partial L_o}{\partial\left(\partial_{\mu_1}\partial_{\mu_2}\phi\right)}\right) = 0, \tag{9}$$

which is the equation of motion with second order Lagrangian.

Similarly, for the third order Lagrangian density $L\left(\phi, \partial_{\mu_1}\phi, \partial_{\mu_1}\partial_{\mu_2}\phi, \partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\phi\right)$ after converting it to first order singular Lagrangian by introducing

$$Z_{\mu_1} = \partial_{\mu_1}\phi, \lambda_{\mu_1} = \left(Z_{\mu_1} - \partial_{\mu_1}\phi\right), \tag{10}$$

$$Z_{\mu_1,\mu_2} = \partial_{\mu_1}\partial_{\mu_2}\phi = \partial_{\mu_2} Z_{\mu_1}, \lambda_{\mu_1,\mu_2} = \left(Z_{\mu_1,\mu_2} - \partial_{\mu_1}\partial_{\mu_2}\phi\right). \tag{11}$$

The new form of the Lagrangian density is written as

$$L\left(\phi, \partial_{\mu_1}\phi, \partial_{\mu_1}\partial_{\mu_2}\phi, \partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\phi\right) = L_o\left(\phi, Z_{\mu_1}, Z_{\mu_1,\mu_2}, \partial_{\mu_3} Z_{\mu_1,\mu_2}\right)$$
$$+\lambda_{\mu_1}\left(Z_{\mu_1} - \partial_{\mu_1}\phi\right) + \lambda_{\mu_1,\mu_2}\left(Z_{\mu_1,\mu_2} - \partial_{\mu_2} Z_{\mu_1}\right). \tag{12}$$

One can make the Legender transformation and determine the corresponding canonical Hamiltonian density of (12), and get

$$H = Z_{\mu_1} P_{\mu_1} + Z_{\mu_1,\mu_2} P_{\mu_1,\mu_2} + \left(\partial_{\mu_3} Z_{\mu_1,\mu_2}\right) P_{\mu_1,\mu_2,\mu_3}$$
$$-L_o\left(\phi, Z_{\mu_1}, Z_{\mu_1,\mu_2}, \partial_{\mu_3} Z_{\mu_1,\mu_2}\right). \tag{13}$$

The corresponding Hamilton's equations of motion can be written as

$$\frac{d\phi}{dx_{\mu_1}} = \frac{\partial H}{\partial P_{\mu_1}} = Z_{\mu_1}, \tag{14}$$

$$\frac{dZ_{\mu_1}}{dx_{\mu_2}} = \frac{\partial H}{\partial P_{\mu_1,\mu_2}} = Z_{\mu_1,\mu_2}, \tag{15}$$

$$\frac{dZ_{\mu_1,\mu_2}}{dx_{\mu_3}} = \frac{\partial H}{\partial P_{\mu_1,\mu_2,\mu_3}} = \partial_{\mu_3} Z_{\mu_1,\mu_2}, \tag{16}$$

$$\frac{dP_{\mu_1}}{dx_{\mu_1}} = -\frac{\partial H}{\partial \phi} = \frac{\partial L_o}{\partial \phi}, \tag{17}$$

$$\frac{dP_{\mu_1,\mu_2}}{dx_{\mu_2}} = -\frac{\partial H}{\partial Z_{\mu_1}} = \frac{\partial L_o}{\partial Z_{\mu_1}} - P_{\mu_1}, \tag{18}$$

$$\frac{dP_{\mu_1,\mu_2,\mu_3}}{dx_{\mu_3}} = -\frac{\partial H}{\partial Z_{\mu_1,\mu_2}} = \frac{\partial L_o}{\partial Z_{\mu_1,\mu_2}} - P_{\mu_1,\mu_2}. \tag{19}$$

One can solve (14-19) simultaneously, and get the equation of motion

$$\frac{\partial L_o}{\partial \phi} - \partial_{\mu_1}\left(\frac{\partial L_o}{\partial\left(\partial_{\mu_1}\phi\right)}\right) + \partial_{\mu_1}\partial_{\mu_2}\left(\frac{\partial L_o}{\partial\left(\partial_{\mu_1}\partial_{\mu_2}\phi\right)}\right)$$
$$-\partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\left(\frac{\partial L_o}{\partial\left(\partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\phi\right)}\right) = 0. \tag{20}$$

Therefore, the equation of motion of higher-order Lagrangian density, with order *n* takes the form

$$\frac{\partial L_o}{\partial \phi} - \partial_{\mu_1}\left(\frac{\partial L_o}{\partial\left(\partial_{\mu_1}\phi\right)}\right) + \cdots + (-1)^n \partial_{\mu_1}\cdots\partial_{\mu_n}\left(\frac{\partial L_o}{\partial\left(\partial_{\mu_1}\cdots\partial_{\mu_n}\phi\right)}\right) = 0. \tag{21}$$

Hamilton-Jacobi Method:

The higher order Lagrangian density can be reduced to first order singular Lagrangian density according to sec. (1)

$$L\left(\phi, \partial_{\mu_1}\phi, \partial_{\mu_1}\partial_{\mu_2}\phi, \partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\phi, \ldots, \partial_{\mu_1}\cdots\partial_{\mu_n}\phi\right) =$$
$$L_o\left(\phi, Z_{\mu_1}, Z_{\mu_1,\mu_2}, \ldots, \partial_{\mu_n} Z_{\mu_1,\ldots,\mu_n}\right) + \lambda_{\mu_1}\left(Z_{\mu_1} - \partial_{\mu_1}\phi\right) + \cdots +$$
$$\lambda_{\mu_1,\ldots,\mu_n}\left(Z_{\mu_1,\ldots,\mu_n} - \partial_{\mu_n} Z_{\mu_1,\ldots,\mu_{n-1}}\right). \tag{22}$$

The canonical momenta of the Lagrangian density (22) reads as

$$P_{\mu_1} = \frac{\partial L}{\partial\left(\partial_{\mu_1}\phi\right)} = -\lambda_{\mu_1}, \tag{23}$$

$$P_{\mu_1,\mu_2} = \frac{\partial L}{\partial\left(\partial_{\mu_2} Z_{\mu_1}\right)} = -\lambda_{\mu_1,\mu_2}, \tag{24}$$

$$P_{\mu_1,\mu_2,\mu_3} = \frac{\partial L}{\partial\left(\partial_{\mu_3} Z_{\mu_1,\mu_2}\right)} = -\lambda_{\mu_1,\mu_2,\mu_3}, \tag{25}$$
$$\vdots$$

$$P_{\mu_1,\ldots,\mu_n} = \frac{\partial L}{\partial\left(\partial_{\mu_n} Z_{\mu_1,\ldots,\mu_{n-1}}\right)} = -\lambda_{\mu_1,\ldots,\mu_n}, \tag{26}$$

$$P_{\mu_1,\ldots,\mu_{n+1}} = \frac{\partial L}{\partial\left(\partial_{\mu_n}\lambda_{\mu_1,\ldots,\mu_n}\right)} = 0. \quad (27)$$

Therefore, the canonical Hamiltonian density can be written as

$$H = Z_{\mu_1}P_{\mu_1} + Z_{\mu_1,\mu_2}P_{\mu_1,\mu_2} + \cdots + Z_{\mu_1,\ldots,\mu_n}P_{\mu_1,\ldots,\mu_n}$$
$$-L_o\left(\phi, Z_{\mu_1}, Z_{\mu_1,\mu_2},\ldots, Z_{\mu_1,\ldots,\mu_n}, \partial_{\mu_n}Z_{\mu_1,\ldots,\mu_n}\right) \quad (28)$$

The set of Hamilton-Jacobi Partial Differential Equations are [9-11]

$$H'_\alpha = P_\alpha + H_\alpha \approx 0, \quad (29)$$
$$H'_{\nu_1} = P_{\nu_1} + \lambda_{\nu_1} \approx 0, \quad (30)$$
$$H'_{\nu_1,\nu_2} = P_{\nu_1,\nu_2} + \lambda_{\nu_1,\nu_2} \approx 0, \quad (31)$$
$$\vdots$$
$$H'_{\nu_1,\ldots,\nu_{n-1}} = P_{\nu_1,\ldots,\nu_{n-1}} + \lambda_{\nu_1,\ldots,\nu_{n-1}} \approx 0, \quad (32)$$
$$H'_{\nu_1,\ldots,\nu_n} = P_{\nu_1,\ldots,\nu_n} + \lambda_{\nu_1,\ldots,\nu_n} \approx 0, \quad (33)$$
$$H'_{\nu_1,\ldots,\nu_{n+1}} = P_{\nu_1,\ldots,\nu_{n+1}} \approx 0. \quad (34)$$

The equations of motion are

$$d\phi = \frac{\partial\left(P_\alpha + H_\alpha\right)}{\partial P_{\mu_1}}dx_\alpha + \frac{\partial\left(P_{\nu_1} + \lambda_{\nu_1}\right)}{\partial P_{\mu_1}}\Big|_{P_{\nu_1}=-\lambda_{\nu_1}} d\phi +$$
$$\frac{\partial\left(P_{\nu_1,\nu_2} + \lambda_{\nu_1,\nu_2}\right)}{\partial P_{\mu_1}}dz_{\nu_1} + \cdots + \frac{\partial\left(P_{\nu_1,\ldots,\nu_n} + \lambda_{\nu_1,\ldots,\nu_n}\right)}{\partial P_{\mu_1}}dZ_{\nu_1,\ldots,\nu_{n-1}}$$
$$+\frac{\partial\left(P_{\nu_1,\ldots,\nu_{n+1}}\right)}{\partial P_{\mu_1}}d\lambda_{\mu_1,\ldots,\mu_n}, \quad (35)$$

$$dZ_{\mu_1} = \frac{\partial\left(P_\alpha + H_\alpha\right)}{\partial P_{\mu_1,\mu_2}}dx_\alpha + \frac{\partial\left(P_{\nu_1} + \lambda_{\nu_1}\right)}{\partial P_{\mu_1,\mu_2}}d\phi +$$
$$\frac{\partial\left(P_{\nu_1,\nu_2} + \lambda_{\nu_1,\nu_2}\right)}{\partial P_{\mu_1,\mu_2}}\Big|_{P_{\nu_1,\nu_2}=-\lambda_{\nu_1,\nu_2}} dZ_{\nu_1} + \cdots + \frac{\partial\left(P_{\nu_1,\ldots,\nu_n} + \lambda_{\nu_1,\ldots,\nu_n}\right)}{\partial P_{\mu_1,\mu_2}}dZ_{\nu_1,\ldots,\nu_{n-1}}$$
$$+\frac{\partial\left(P_{\nu_1,\ldots,\nu_{n+1}}\right)}{\partial P_{\mu_1,\mu_2}}d\lambda_{\mu_1,\ldots,\mu_n}, \quad (36)$$
$$\vdots$$

$$dZ_{\mu_1,\ldots,\mu_{n-1}} = \frac{\partial\left(P_\alpha + H_\alpha\right)}{\partial P_{\mu_1,\ldots,\mu_n}} dx_\alpha + \frac{\partial\left(P_{\nu_1} + \lambda_{\nu_1}\right)}{\partial P_{\mu_1,\ldots,\mu_n}} d\phi +$$
$$\frac{\partial\left(P_{\nu_1,\nu_2} + \lambda_{\nu_1,\nu_2}\right)}{\partial P_{\mu_1,\ldots,\mu_n}} dZ_{\nu_1} + \cdots + \frac{\partial\left(P_{\nu_1,\ldots,\nu_n} + \lambda_{\nu_1,\ldots,\nu_n}\right)}{\partial P_{\mu_1,\ldots,\mu_n}} dZ_{\nu_1,\ldots,\nu_{n-1}}$$
$$+ \frac{\partial\left(P_{\nu_1,\ldots,\nu_{n+1}}\right)}{\partial P_{\mu_1,\ldots,\mu_n}} d\lambda_{\mu_1,\ldots,\mu_n}, \qquad (37)$$

$$dP_{\mu_1} = -\frac{\partial\left(P_\alpha + H_\alpha\right)}{\partial\phi} dx_\alpha - \frac{\partial\left(P_{\nu_1} + \lambda_{\nu_1}\right)}{\partial\phi} d\phi - \frac{\partial\left(P_{\nu_1,\nu_2} + \lambda_{\nu_1,\nu_2}\right)}{\partial\phi} dZ_{\nu_1}$$
$$-\cdots - \frac{\partial\left(P_{\nu_1,\ldots,\nu_n} + \lambda_{\nu_1,\ldots,\nu_n}\right)}{\partial\phi} dZ_{\nu_1,\ldots,\nu_{n-1}} - \frac{\partial\left(P_{\nu_1,\ldots,\nu_{n+1}}\right)}{\partial\phi} d\lambda_{\mu_1,\ldots,\mu_n}, \qquad (38)$$

$$dP_{\mu_1,\mu_2} = -\frac{\partial\left(P_\alpha + H_\alpha\right)}{\partial Z_{\mu_1}} dx_\alpha - \frac{\partial\left(P_{\nu_1} + \lambda_{\nu_1}\right)}{\partial Z_{\mu_1}} d\phi - \frac{\partial\left(P_{\nu_1,\nu_2} + \lambda_{\nu_1,\nu_2}\right)}{\partial Z_{\mu_1}} dZ_{\nu_1}$$
$$-\cdots - \frac{\partial\left(P_{\nu_1,\ldots,\nu_n} + \lambda_{\nu_1,\ldots,\nu_n}\right)}{\partial Z_{\mu_1}} dZ_{\nu_1,\ldots,\nu_{n-1}} - \frac{\partial\left(P_{\nu_1,\ldots,\nu_{n+1}}\right)}{\partial Z_{\mu_1}} d\lambda_{\mu_1,\ldots,\mu_n}, \qquad (39)$$

$$dP_{\mu_1,\mu_2,\mu_3} = -\frac{\partial\left(P_\alpha + H_\alpha\right)}{\partial Z_{\mu_1,\mu_2}} dx_\alpha - \frac{\partial\left(P_{\nu_1} + \lambda_{\nu_1}\right)}{\partial Z_{\mu_1,\mu_2}} d\phi - \frac{\partial\left(P_{\nu_1,\nu_2} + \lambda_{\nu_1,\nu_2}\right)}{\partial Z_{\mu_1,\mu_2}} dZ_{\nu_1}$$
$$-\cdots - \frac{\partial\left(P_{\nu_1,\ldots,\nu_n} + \lambda_{\nu_1,\ldots,\nu_n}\right)}{\partial Z_{\mu_1,\mu_2}} dZ_{\nu_1,\ldots,\nu_{n-1}} - \frac{\partial\left(P_{\nu_1,\ldots,\nu_{n+1}}\right)}{\partial Z_{\mu_1,\mu_2}} d\lambda_{\mu_1,\ldots,\mu_n}, \qquad (40)\vdots$$

$$dP_{\mu_1,\ldots,\mu_n} = -\frac{\partial\left(P_\alpha + H_\alpha\right)}{\partial Z_{\mu_1,\ldots,\mu_{n-1}}} dx_\alpha - \frac{\partial\left(P_{\nu_1} + \lambda_{\nu_1}\right)}{\partial Z_{\mu_1,\ldots,\mu_{n-1}}} d\phi - \frac{\partial\left(P_{\nu_1,\nu_2} + \lambda_{\nu_1,\nu_2}\right)}{\partial Z_{\mu_1,\ldots,\mu_{n-1}}} dZ_{\nu_1}$$
$$-\cdots - \frac{\partial\left(P_{\nu_1,\ldots,\nu_n} + \lambda_{\nu_1,\ldots,\nu_n}\right)}{\partial Z_{\mu_1,\ldots,\mu_{n-1}}} dZ_{\nu_1,\ldots,\nu_{n-1}} - \frac{\partial\left(P_{\nu_1,\ldots,\nu_{n+1}}\right)}{\partial Z_{\mu_1,\ldots\mu_{n-1}}} d\lambda_{\mu_1,\ldots,\mu_n}, \qquad (41)$$

$$dP_{\mu_1,\ldots,\mu_{n+1}} = -\frac{\partial\left(P_\alpha + H_\alpha\right)}{\partial\lambda_{\mu_1,\ldots,\mu_n}} dx_\alpha - \frac{\partial\left(P_{\nu_1} + \lambda_{\nu_1}\right)}{\partial\lambda_{\mu_1,\ldots,\mu_n}} d\phi - \frac{\partial\left(P_{\nu_1,\nu_2} + \lambda_{\nu_1,\nu_2}\right)}{\partial\lambda_{\mu_1,\ldots,\mu_n}} dZ_{\nu_1}$$
$$-\cdots - \frac{\partial\left(P_{\nu_1,\ldots,\nu_n} + \lambda_{\nu_1,\ldots,\nu_n}\right)}{\partial\lambda_{\mu_1,\ldots,\mu_n}} dZ_{\nu_1,\ldots,\nu_{n-1}} - \frac{\partial\left(P_{\nu_1,\ldots,\nu_{n+1}}\right)}{\partial\lambda_{\mu_1,\ldots\mu_n}} d\lambda_{\mu_1,\ldots,\mu_n}. \qquad (42)$$

The equations (35-42) are reduced to

$$d\phi = Z_{\mu_1} dx_{\mu_1}, \tag{43}$$

$$dZ_{\mu_1} = Z_{\mu_1,\mu_2} dx_{\mu_2}, \tag{44}$$

$$dZ_{\mu_1,\mu_2} = Z_{\mu_1,\mu_2,\mu_3} dx_{\mu_3}, \tag{45}$$

$$\vdots$$

$$dZ_{\mu_1,\ldots,\mu_{n-1}} = \left(\partial_{\mu_n} Z_{\mu_1,\ldots,\mu_n}\right) dx_{\mu_n}, \tag{46}$$

$$dP_{\mu_1} = \frac{\partial L_o}{\partial \phi} dx_{\mu_1}, \tag{47}$$

$$dP_{\mu_1,\mu_2} = \left(\frac{\partial L_o}{\partial Z_{\mu_1}} - P_{\mu_1}\right) dx_{\mu_2}, \tag{48}$$

$$\vdots$$

$$dP_{\mu_1,\ldots,\mu_n} = \left(\frac{\partial L_o}{\partial Z_{\mu_1,\ldots,\mu_{n-1}}} - P_{\mu_1,\ldots,\mu_{n-1}}\right) dx_{\mu_n}, \tag{49}$$

$$dP_{\mu_1,\ldots,\mu_{n+1}} = 0. \tag{50}$$

The equations (43-50) are integrable if and only if the integrability conditions are satisfied. That is the variation of relations (29-34) respectively are

$$dH'_{\alpha} = dP_{\alpha} + dH_{\alpha} \approx 0 \tag{51}$$

$$dH'_{\nu_1} = dP_{\nu_1} + d\lambda_{\nu_1} \approx 0, \tag{52}$$

$$dH'_{\nu_1,\nu_2} = dP_{\nu_1,\nu_2} + d\lambda_{\nu_1,\nu_2} \approx 0, \tag{53}$$

$$\vdots$$

$$dH'_{\nu_1,\ldots,\nu_{n-1}} = dP_{\nu_1,\ldots,\nu_{n-1}} + d\lambda_{\nu_1,\ldots,\nu_{n-1}} \approx 0, \tag{54}$$

$$dH'_{\nu_1,\ldots,\nu_n} = dP_{\nu_1,\ldots,\nu_n} + d\lambda_{\nu_1,\ldots,\nu_n} \approx 0, \tag{55}$$

$$dH'_{\nu_1,\ldots,\nu_{n+1}} = dP_{\nu_1,\ldots,\nu_{n+1}} \approx 0. \tag{56}$$

Relations (41-56) are satisfied under the following conditions:

$$\frac{d}{dx_{\nu_1}} \lambda_{\nu_1} = -\frac{\partial L_o}{\partial \phi}, \tag{57}$$

$$\frac{d}{dx_{\nu_2}} \lambda_{\nu_1,\nu_2} = -\frac{\partial L_o}{\partial Z_{\nu_1}} + P_{\nu_1}, \tag{58}$$

$$\vdots$$

$$\frac{d}{dx_{\nu_{n-1}}}\lambda_{\nu_1,\ldots,\nu_{n-1}} = P_{\nu_1,\ldots,\nu_{n-2}} - \frac{\partial L_o}{\partial Z_{\nu_1,\ldots,\nu_{n-2}}}, \qquad (59)$$

$$\frac{d}{dx_{\nu_n}}\lambda_{\nu_1,\ldots,\nu_n} = P_{\nu_1,\ldots,\nu_{n-1}} - \frac{\partial L_o}{\partial Z_{\nu_1,\ldots,\nu_{n-1}}}, \qquad (60)$$

$$P_{\nu_1,\ldots,\nu_n} = \frac{\partial L_o}{\partial Z_{\nu_1,\ldots,\nu_n}}. \qquad (61)$$

Solving (43-50) simultaneously, one obtain

$$\frac{\partial L_o}{\partial \phi} - \partial_{\mu_1}\left(\frac{\partial L_o}{\partial Z_{\mu_1}}\right) + \partial_{\mu_1}\partial_{\mu_2}\left(\frac{\partial L_o}{\partial Z_{\mu_1,\mu_2}}\right) + \cdots +$$

$$(-1)^n \partial_{\mu_1}\cdots\partial_{\mu_n}\left(\frac{\partial L_o}{\partial Z_{\mu_1,\ldots,\mu_{n-1}}}\right) = 0, \qquad (62)$$

OR

$$\frac{\partial L_o}{\partial \phi} - \partial_{\mu_1}\left(\frac{\partial L_o}{\partial(\partial_{\mu_1}\phi)}\right) + \partial_{\mu_1}\partial_{\mu_2}\left(\frac{\partial L_o}{\partial(\partial_{\mu_1}\partial_{\mu_2}\phi)}\right) + \cdots +$$

$$(-1)^n \partial_{\mu_1}\cdots\partial_{\mu_n}\left(\frac{\partial L_o}{\partial(\partial_{\mu_1}\cdots\partial_{\mu_n}\phi)}\right). \qquad (63)$$

which is the same form of (21).

An example:

As an example, let us consider the effective Lagragian [11]

$$L_{eff} = L_o + \varepsilon L_I = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2}M^2\varphi^2$$

$$+\varepsilon L_I\left(\varphi, \partial^\mu\varphi, \ldots, \partial^{\mu_1}\cdots\partial^{\mu_n}\varphi\right), \qquad (64)$$

where L_o represents a free-massive Klein-Gordon theory and L_I contains the effective interactions which depends on the derivatives of the scalar fields up to order (n). These interaction are governed by the coupling constant with $\varepsilon \ll 1$.

To convert the higher order Interaction Lagrangian density to first-order, let

$$Z^{\mu_1} = \partial^{\mu_1}\varphi, \alpha^{\mu_1} = Z^{\mu_1} - \partial^{\mu_1}\varphi, \qquad (65)$$

$$Z^{\mu_1,\mu_2} = \partial^{\mu_1}\partial^{\mu_2}\varphi = \partial^{\mu_2}Z^{\mu_1}, \alpha^{\mu_1,\mu_2} = Z^{\mu_1,\mu_2} - \partial^{\mu_2}Z^{\mu_1}, \qquad (66)$$

$$Z^{\mu_1,\mu_2,\mu_3} = \partial^{\mu_3}Z^{\mu_1,\mu_2}, \alpha^{\mu_1,\mu_2,\mu_3} = Z^{\mu_1,\mu_2,\mu_3} - \partial^{\mu_3}Z^{\mu_1,\mu_2}, \qquad (67)\vdots$$

$$Z^{\mu_1,\dots,\mu_n} = \partial^{\mu_n} Z^{\mu_1,\dots,\mu_{n-1}}, \alpha^{\mu_1,\dots,\mu_n} = Z^{\mu_1,\dots,\mu_n} - \partial^{\mu_n} Z^{\mu_1,\dots,\mu_{n-1}}. \qquad (68)$$

Therefore, the new first order singular Lagrangian density takes the form

$$L_{red} = \frac{1}{2}\left(\partial^{\nu}\varphi g_{\mu\nu}\right)\left(\partial^{\mu}\varphi\right) - \frac{1}{2}M^{2}\varphi^{2} + \varepsilon L_{oI}\left(\varphi, Z^{\mu_1}, Z^{\mu_1,\mu_2}, \dots, Z^{\mu_1,\dots,\mu_n}\right)$$
$$+\varepsilon\alpha^{\mu_1}\left(Z^{\mu_1} - \partial^{\mu_1}\varphi\right) + \cdots + \varepsilon\alpha^{\mu_1,\dots,\mu_n}\left(Z^{\mu_1,\dots,\mu_n} - \partial^{\mu_1} Z^{\mu_1,\dots,\mu_{n-1}}\right), \qquad (69)$$

where $g_{\mu\nu} = diag\left(+1,-1,-1,-1\right)$, the metric tensor.

The canonical momenta are

$$P^{\mu_1} = \frac{\partial L_{eff}}{\partial\left(\partial^{\mu_1}\varphi\right)} = \partial^{\mu_1}\varphi - \varepsilon\alpha^{\mu_1}, \qquad (70)$$

$$P^{\mu_1,\mu_2} = \frac{\partial L_{eff}}{\partial\left(\partial^{\mu_2} Z^{\mu_1}\right)} = -\varepsilon\alpha^{\mu_1,\mu_2}, \qquad (71)$$

$$P^{\mu_1,\mu_2,\mu_3} = \frac{\partial L_{eff}}{\partial\left(\partial^{\mu_3} Z^{\mu_1,\mu_2}\right)} = -\varepsilon\alpha^{\mu_1,\mu_2,\mu_3}, \qquad (72)\vdots$$

$$P^{\mu_1,\dots,\mu_n} = \frac{\partial L_{eff}}{\partial\left(\partial^{\mu_n} Z^{\mu_1,\dots,\mu_{n-1}}\right)} = -\varepsilon\alpha^{\mu_1,\dots,\mu_n}, \qquad (73)$$

$$P^{\mu_1,\dots,\mu_{n+1}} = \frac{\partial L_{eff}}{\partial\left(\partial^{\mu_n}\alpha^{\mu_1,\dots,\mu_n}\right)} = 0. \qquad (74)$$

Thus, the canonical Hamiltonian density is

$$H = -\frac{1}{2}\left(\partial^{\mu_1}\varphi\right)\left(\partial_{\mu_1}\varphi\right) + \frac{1}{2}M^{2}\varphi^{2} + Z^{\mu_1}P^{\mu_1} + Z^{\mu_1,\mu_2}P^{\mu_1,\mu_2} + \cdots +$$
$$Z^{\mu_1,\dots,\mu_n}P^{\mu_1,\dots,\mu_n} - \varepsilon L_{oI}\left(\varphi, Z^{\mu_1}, Z^{\mu_1,\mu_2}, \dots, \partial^{\mu_n} Z^{\mu_1,\dots,\mu_{n-1}}\right). \qquad (75)$$

The set of Hamilton-Jacobi Partial Differential Equations are

$$H'^{\alpha} = P^{\alpha} + H^{\alpha} = 0, \qquad (76)$$
$$H'^{\mu_1} = P^{\mu_1} + \varepsilon\alpha^{\mu_1} - \partial^{\mu_1}\varphi = 0, \qquad (77)$$
$$H'^{\mu_1,\mu_2} = P^{\mu_1,\mu_2} + \varepsilon\alpha^{\mu_1,\mu_2} = 0, \qquad (78)\vdots$$
$$H'^{\mu_1,\dots,\mu_n} = P^{\mu_1,\dots,\mu_n} + \varepsilon\alpha^{\mu_1,\dots,\mu_n} = 0, \qquad (79)$$
$$H'^{\mu_1,\dots,\mu_{n+1}} = P^{\mu_1,\dots,\mu_{n+1}} = 0. \qquad (80)$$

The equations of motion are

$$d\varphi = Z^{\mu_1}dx^{\mu_1}, \qquad (81)$$
$$dZ^{\mu_1} = Z^{\mu_1,\mu_2}dx^{\mu_2}, \qquad (82)$$
$$dZ^{\mu_1,\mu_2} = Z^{\mu_1,\mu_2,\mu_3}dx^{\mu_3}, \qquad (83)\vdots$$

$$dZ^{\mu_1,\ldots,\mu_{n-1}} = Z^{\mu_1,\ldots,\mu_n} dx^{\mu_n}, \tag{84}$$

$$dZ^{\mu_1,\ldots,\mu_n} = \left(\partial^{\mu_n} Z^{\mu_1,\ldots,\mu_n}\right) dx^{\mu_n}, \tag{85}$$

$$dP^{\mu_1} = \left[\varepsilon \frac{\partial L_{oI}}{\partial \varphi} - M^2 \varphi\right] dx^{\mu_1}, \tag{86}$$

$$dP^{\mu_1,\mu_2} = \left[\partial_{\mu_1}\varphi + \varepsilon \frac{\partial L_{oI}}{\partial Z^{\mu_1}} - P^{\mu_1}\right] dx^{\mu_2}, \tag{87}$$

$\vdots$

$$dP^{\mu_1,\ldots,\mu_{2n-1}} = \left[\varepsilon \frac{\partial L_{oI}}{\partial Z^{\mu_{n-2}}} - P^{\mu_1,\ldots,\mu_{n-2}}\right] dx^{\mu_{n-1}}, \tag{88}$$

$$dP^{\mu_1,\ldots,\mu_n} = \left[\varepsilon \frac{\partial L_{oI}}{\partial Z^{\mu_1,\ldots,\mu_{n-1}}} - P^{\mu_1,\ldots,\mu_{n-1}}\right] dx^{\mu_n}, \tag{89}$$

$$dP^{\mu_1,\ldots,\mu_{n+1}} = 0. \tag{90}$$

These equations are integrable if and only if the variations of (76-80) vanish

$$dH'^{\alpha} = dP^{\alpha} + dH^{\alpha} = 0, \tag{91}$$

$$dH'^{\mu_1} = dH^{\mu_1} + \varepsilon d\alpha^{\mu_1} - d\left(\partial^{\mu_1}\varphi\right) = 0, \tag{92}$$

$$dH'^{\mu_1,\mu_2} = dP^{\mu_1,\mu_2} + \varepsilon d\alpha^{\mu_1,\mu_2} = 0, \tag{93}$$

$\vdots$

$$dH'^{\mu_1,\ldots,\mu_n} = dP^{\mu_1,\ldots,\mu_n} + \varepsilon d\alpha^{\mu_1,\ldots,\mu_n} = 0, \tag{94}$$

$$dH'^{\mu_1,\ldots,\mu_{n+1}} = dP^{\mu_1,\ldots,\mu_{n+1}} = 0. \tag{95}$$

The equation (91) vanishes identically, but the equations (92-95) are vanish under the following conditions

$$d\alpha^{\mu_1} = -\frac{1}{\varepsilon}\left[\left(\varepsilon \frac{\partial L_{oI}}{\partial \varphi} - M^2\varphi\right) dx^{\mu_1} - Z^{\mu_1,\mu_2} dx^{\mu_2}\right], \tag{96}$$

$$d\alpha^{\mu_1,\mu_2} = \frac{1}{\varepsilon}\left[P^{\mu_1} - \left(\partial_{\mu_1}\varphi\right) - \varepsilon \frac{\partial L_{oI}}{\partial Z^{\mu_1}}\right] dx^{\mu_2}, \tag{97}$$

$\vdots$

$$d\alpha^{\mu_1,\ldots,\mu_n} = \left[\frac{1}{\varepsilon} P^{\mu_1,\ldots,\mu_{n-1}} - \frac{\partial L_{oI}}{\partial Z^{\mu_1,\ldots,\mu_{n-1}}}\right] dx^{\mu_n}, \tag{98}$$

$$P^{\mu_1,\ldots,\mu_n} = \varepsilon \frac{\partial L_{oI}}{\partial Z^{\mu_1,\ldots,\mu_n}}. \tag{99}$$

One can solve (81-90) simultaneously to obtain

$$M^2\varphi + \partial^{\mu_1}\partial_{\mu_1}\varphi - \varepsilon\left[\frac{\partial L_{oI}}{\partial \varphi} - \partial_{\mu_1}\left(\frac{\partial L_{oI}}{\partial\left(\partial^{\mu_1}\varphi\right)}\right) + \cdots +\right.$$

$$\left(-1\right)^{n} \partial^{\mu_1} \cdots \partial^{\mu_n} \left(\frac{\partial L_{oI}}{\partial\left(\partial^{\mu_1} \cdots \partial^{\mu_n} \varphi\right)}\right)\Bigg] = 0. \qquad (100)$$

This result is in agreement with the results obtained in Ref. [3].

CONCLUSION:

In this paper, we have studied the Hamilton-Jacobi method for higher order Lagrangian density by treating them as first order Lagrangians with constraints. This gives equation (63) which agrees with the result obtained in equation (21). The *effective* higher-order Lagrangian [9] was studied as an example.

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